Non-local means theory based Perona–Malik model for image denoising

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A B S T R A C T

Among various kinds of image denoising methods, the Perona–Malik model is a representative Partial Differential Equation based (PDE-based) algorithm which effectively removes the noise as well as having edge enhancement simultaneously through anisotropic diffusion controlled by the diffusion coefficient. However, the unstable behavior of the Perona–Malik model introduces staircasing artifacts in the processed images. To realize less diffusion in the texture region and to get more smooth in flat region while implementing image denoising, we propose an improved Perona–Malik model based on non-local means theory, which assumes that the image contains an extensive amount of self-similarity and uses the similarity between the region around the center pixel and the region outside the center pixel to give a more reasonable description of the image. The improved algorithm is applied on numerical simulation and practical images, and the quantitative analyzing results prove that the modified anisotropic diffusion model can preserve textures effectively while ruling out the noise, meanwhile, the staircasing effects are decreased obviously.

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1. Introduction

Image denoising is a fundamental step for image segmentation, edge detection and feature extraction \cite{1,2,3,4,5,6,7,8}. In many practical cases, due to the high frequency characteristics and useful detailed information of noise, it is difficult to reserve edge or texture information efficiently while ruling out image noise for most of the image denoising methods \cite{9,10,11,12,13,14,15}. The basic principles of the denoising algorithms can be summarized to less blur in the texture region and more smooth in flat region \cite{16,17}. According to the principles, the Partial Differential Equation based (PDE-based) image denoising methods were developed and applied in computer visualization broadly. Among them, the representative one is the nonlinear diffusion approach named P–M model proposed by Perona and Malik \cite{20,21,22}. Through anisotropic diffusion controlled by the diffusion coefficient which is small while the gradient of image is large, the P–M model can effectively remove the noise as well as contain edge enhancement simultaneously. However, PDE-based denoising algorithm has inherent drawbacks and then various modified approaches were proposed. Wu et al. \cite{23} pointed out that PDE had “staircasing” in the processed image and textures could not be well preserved. They developed a hybrid image denoising algorithm which combined the adaptive PDE with wavelet shrinkage to eliminate the above drawbacks. Selim \cite{24} also mentioned that the staircasing phenomenon was raised by unstable behavior of the P–M model and investigated how the PDEs of the P–M model could be given an existence and uniqueness resolution. Shih et al. \cite{25} figured out that the nonlinear diffusion model was weak in removing salt-and-pepper noise and they proposed a convection–diffusion filter by adding a convection term in the modified diffusion equation as a physical interpretation for denoising.

In this paper, we propose an improved Perona-Malik model which is based on non-local means theory (NL P–M model) successfully used for image denoising and detailed information preserving together. Non-local means theory assumes that the image contains an extensive amount of self-similarity and uses the similarity between the region around the center pixel and the region outside the center pixel to give a more reasonable description of the image \cite{26}. So the true intensity in each pixel can be estimated more accurately \cite{27}. Considering the remarkable performance of non-local means theory, it is applied to the...
P–M model so as to eliminate the staircasing effects and improve the performance of texture preserving with an ideal stability.

In the following section, the derivation process of the P–M model is firstly introduced and its instability is analyzed. In Section 3, the modified P–M model based on non-local means theory (NL–P–M model) is developed. Then the NL–P–M model is employed in Lena, phantom and human brain CT images to verify its virtues of decreasing image noise while preserving the detailed information effectively. Furthermore, quantitative comparison proves that the improved model can improve the performance of texture preserving with an ideal stability. Finally, the conclusion is given at the last section.

2. Instability analysis of P–M model

The nonlinear diffusion equation is developed to create a sequence of continuous images \(u(x,y,t)\) on the abstract scale \(t\) and through the nonlinear diffusion equation to remove the noise during the scaling \("time\) revolution [17]. The iteration expression of P–M model is

\[
\begin{align*}
\{ (\partial u / \partial t) &= \text{div} g(|\nabla u|) \nabla u \quad \text{for} \quad t \in (0, T) \\
u(x,y,t)|_{t=0} &= u_0(x,y)
\end{align*}
\]  
(1)

where, \(\nabla u, |\nabla u|\) are the image gradient and \(\nabla u = [u_x, u_y]\), \(|\nabla u| = \sqrt{u_x^2 + u_y^2}\), \(u_x\) and \(u_y\) are the first-order directional derivatives of \(u(x,y,t)\). \(u(x,y,t)\) is the processed result, \(u_0(x,y)\) is the given noised image, \(g(|\nabla u|)\) is the diffusion coefficient which controls the diffusion speed. The ideal diffusion process is supposed to diffuse quickly in smooth region and diffuse slowly or even stop in the region of edge. Correspondingly, we have

\[
\begin{align*}
\lim_{|\nabla u| \to 0} g(|\nabla u|) &= 1, \quad \text{fast diffusion} \\
\lim_{|\nabla u| \to \infty} g(|\nabla u|) &= 0, \quad \text{slow or stopped diffusion}
\end{align*}
\]  
(2)

Based on the above analysis, there are two different forms of \(g(|\nabla u|)\) as follows:

\[
g(|\nabla u|) = \frac{1}{1 + (|\nabla u|/k)^2}, \quad g(|\nabla u|) = e^{-|\nabla u|/\xi^2}
\]  
(3)

where, \(k\) is the threshold used for distinguishing edge region from smooth region. As a result, we can draw the conclusion that the process of the P–M denoising method is actually to solve the PDE as shown below. \(\eta\) and \(\xi\) are defined as the orthogonal directional gradients:

\[
\eta = \frac{\nabla u}{|\nabla u|} = \left(\frac{u_x}{u_x^2 + u_y^2}, \frac{u_y}{u_x^2 + u_y^2}\right), \quad \xi = \frac{\nabla^2 u}{|\nabla u|} = \left(\frac{u_{xx} u_x + 2u_x u_y u_y + u_{yy} u_y}{u_x^2 + u_y^2}, \frac{u_{xx} u_x + 2u_x u_y u_y + u_{yy} u_y}{u_x^2 + u_y^2}\right)
\]  
(4)

Then we can get:

\[
u_t = \frac{\partial u}{\partial \eta} = \left(\frac{u_x}{u_x^2 + u_y^2}, \frac{u_y}{u_x^2 + u_y^2}\right) \cdot \left(\frac{u_x}{u_x^2 + u_y^2}, \frac{u_y}{u_x^2 + u_y^2}\right) = \sqrt{u_x^2 + u_y^2} = |\nabla u|
\]  
(5)

and

\[
u_{\eta\eta} = \frac{\partial u}{\partial \xi} = \left[\frac{\partial u_x}{\partial \xi} \frac{\partial u_y}{\partial \xi}\right] = \left(\frac{u_x}{u_x^2 + u_y^2}, \frac{u_y}{u_x^2 + u_y^2}\right) \cdot \left(\frac{u_{xx} u_x + 2u_x u_y u_y + u_{yy} u_y}{u_x^2 + u_y^2}, \frac{u_{xx} u_x + 2u_x u_y u_y + u_{yy} u_y}{u_x^2 + u_y^2}\right)
\]  
(6)

where, \(u_t\) and \(u_{\eta\eta}\) are the first-order and the second-order directional derivatives of \(u(x,y,t)\) respectively. \(u_{xx}, u_{yy}\) and \(u_{xy}\) are the second-order directional derivatives of \(u(x,y,t)\). Similarly, the second-order derivative of \(\xi\) can be derived as

\[
u_{\eta\eta} = \frac{u_{xx}^2 u_x + 2u_x u_y u_{yy} + u_{yy}^2 u_y}{u_x^2 + u_y^2} = \frac{u_{xy}^2}{u_x^2 + u_y^2} + \frac{u_{xx}^2}{u_x^2 + u_y^2} + \frac{u_{yy}^2}{u_x^2 + u_y^2}
\]  
(7)

From Eq. (1), the following solution can be obtained,

\[
\frac{\partial u}{\partial t} = \text{div} g(|\nabla u|) \nabla u
\]  
(8)

\[
= |g(|\nabla u|)| u_x + \frac{\partial g(|\nabla u|)}{\partial |\nabla u|} \nabla u
\]  
(9)

\[
= g(|\nabla u|) \left(\frac{\partial |\nabla u|}{\partial u_x} u_x + \frac{\partial |\nabla u|}{\partial u_y} u_y\right) + g(|\nabla u|)(u_{xx} + u_{yy})
\]  
(10)

\[
= g(|\nabla u|) \left(\frac{u_{xx}^2 u_x + 2u_x u_y u_{yy} + u_{yy}^2 u_y}{u_x^2 + u_y^2}ight) + g(|\nabla u|)(u_{xx} + u_{yy})
\]  
(11)

Obviously, \(g(|\nabla u|)\) in Eq. (9) is always positive which means diffusing forward and making image smooth. However, \(g(|\nabla u|)|\nabla \| + g(|\nabla u|)\) can be either positive or negative leading to unstable results. Jeny et al. [18] proposed another version of P–M model based on the mean curvature equation (MCE) which omitted \(g(|\nabla u|)|\nabla \| + g(|\nabla u|)\) and left \(u_0=|\nabla u|\).

We derive \(\nabla u\) in \(3 \times 3\) local region centered at pixel \((i,j)\) as follows:

\[
\begin{align*}
\nabla u_{ij} &= u_{i+1,j} - u_{i-1,j} \\
\nabla u_{ij} &= u_{i,j+1} - u_{i,j-1} \\
\nabla u_{ij} &= u_{i+1,j+1} - u_{i-1,j-1} \\
\nabla u_{ij} &= u_{i-1,j+1} - u_{i+1,j-1}
\end{align*}
\]  
(12)

So we have

\[
|\nabla u| = |\nabla u_{ij}| + |\nabla u_{ij}| + |\nabla u_{ij}| + |\nabla u_{ij}|
\]  
(13)

Combining Eq. (1) with (11), the discrete iteration of \(n + 1\) times of the P–M model can be expressed:

\[
u_{n+1} = \nu_n + \Delta t \text{div} g(|\nabla u|)\nabla u
\]  
(14)

where, \(\Delta t\) is the “time” step which indicates the diffusion speed and the suggested value is \(0.05 \leq \Delta t \leq 0.5\). Larger \(\Delta t\) and iteration cycles can lead smoother image at the cost of losing detailed information. Actually, \(\Delta t\) and iteration cycles should be set reasonably.

By further analysis, the Eq. (12) is still not a stable process, which means when we apply it to noisy image, more than one local optimal point would appear on the whole image and the result image may be distorted by “staircasing” [22]. The diffusion coefficient of the P–M model is just determined by gradient information which only reflects the relationship among adjacent single pixels. Due to the finite information in a single pixel, usually the content of the information is occupied by noise, the calculated diffusion coefficient of the noised image deviates from its true value of ideal image. How to combine the diffusion coefficient with the image features accurately is the core solution to avoid “staircasing” artifacts.

3. Development of modified P–M model

Using the neighborhood similarity of pixels is a feasible approach to extract features from noisy image and then non-local means theory is proposed [25–28]. This theory assumes that
the image contains an extensive amount of self-similarity and uses the similarity between the region around the center pixel and the region outside the center pixel to get a more reasonable description of the image. According to the theory, the true intensity in the image pixel number, keeps the details clearer. The improved version of Eq.(12) is shown as

$$u_{n+1}^{NL} = u_{n}^{NL} + \Delta t g((\nabla u_i)|\nabla u_i).$$

(17)

Here we define this new model as NL-P-M model. The cost in computing of the NL-P-M model increases with iteration times and the non-local window size. Before using this new model to remove image noise, there are three variables to set: the edge threshold $k$, the standard deviation of Gaussian kernel $\alpha$ and characteristic parameter of image noise $h$. It can be deduced that when we set the non-local window size as $1 \times 1$, NL-P-M model becomes P-M model. On the other hand, if we iterate only once and $\Delta t = 0$, the NL-P-M model becomes standard NL-means algorithm.

From the above discussion, NL-P-M model is improved by the five-step denoising method, as follows:

(a) Conduct non-local means filtering upon the current pixel $(x, y)$ to get $u_{n}^{NL}$ according to Eq. (13).
(b) Calculate image gradient along right, left, up, and down directions according to Eq. (16).
(c) Calculate $g_l(\nabla u_l) \cdot |\nabla u_l|$ according to Eq. (11).
(d) Restore the noisy pixel using Eq. (17).
(e) Repeat (a)-(d) on the next pixel until the final pixel in the image.

4. Results

To verify the feasibility of the modified P-M method, Lena image, human brain CT image and Phantom image are employed to compare the denoising results. The employed gray-leveled images are of same size and bit depth of $256 \times 256$ and 8-bits. Fig. 1(a) shows the four images containing Gauss-type noise with the mean value of zero and standard deviation of $\sigma = 20$. Fig. 1(b)-(e) are the denoised results based on this three models respectively. The denoising parameters are shown as follows:

P-M model: iteration 20 times with edge threshold $k = 33$
NL means model: $K = 5$, $L = 15$, $a = 50$ and $h = 120$
NL P-M model: iteration 2 times with $K = 5$, $L = 15$, edge threshold $k = 33$, $a = 50$ and $h = 120$

where, $K$ and $L$ are the non-local window size.

To obtain an objective evaluation of the improved model, we applied Peak Signal to Noise Ratio (PSNR) and the mean Structural Similarity (MSSIM) to assess the denosing and useful information preserving ability of the modified model [30]. Table 1 shows the value of PSNR and MSSIM of Lena and Phantom images with different noise level. PSNR is used to measure the noise reduction effect and its larger value indicates the smoother processed image. MSSIM is a method for measuring the similarity between an initial or distortion-free image and processed image. From the statistical results in Table 1, all the three models perform well in noise restraining and make the noisy image smoother. But in comparison, “staircasing” artifacts are contained in the P-M model processed image as analyzed before, non-local means (NL-means) model does not induce obvious artifacts, so its PSNR value is larger than the P-M model and even sometimes larger than the NL-P-M model, but it blurs the image as shown in Fig. 1, and it has a smaller MSSIM value than the P-M model. The modified model (NL-P-M model) integrates the virtues of the P-M and NL-means models, so the texture and detailed information are much better preserved while the noise is decreased effectively than the other two models; so the NL-P-M model processed image has the largest value of MSSIM and its PSNR value keeps in the same level with the other two models as shown in Table 1.

At the same time, we also applied the P-M model and NL-P-M model to cross-sectional image of a cigarette scanned by Micro-CT. The size of CT image is $1024 \times 1024$, and its bit depth is 16-bits. The original CT image is shown in Fig. 2(a). Fig. 2(b) and (c) shows the denoised results by these two methods. The denoising parameters are shown as follows:

P-M model: iteration 27 times with edge threshold $k = 35$
NL-P-M model: iteration 6 times with $K = 5$, $L = 11$, edge threshold $k = 35$, $a = 75$ and $h = 900$

The curves of the same line in the three images are plotted in Fig. 2(d), which further prove that NL-P-M model is an ideal method to restore and enhance such kind of images with low signal to noise ratio (SNR). Also, we process all cross-sectional images of one part of a cigarette by the P-M model and NL-P-M model, then load the two kinds of denoised image sequences into Mimics10 (Materialise, Belgium) to create 3D virtual model of the cigarette. Artifacts and blur exit in the 3D virtual model due to the limitation of P-M method as shown in Fig. 2(e). As the improvement of the P-M model, the modified model (NL-P-M model) provides the 3D virtual cigarette much smoother outside surface and much clearer inner structure as shown in Fig. 2(f).

5. Conclusions

In this paper, the drawbacks of the P-M model upon its partial differential equation have been analyzed and an improved denoising hybrid model has been proposed to avoid these drawbacks. The improved model introduces the non-local means theory and keeps the advantages of the P-M and non-local means models. The improved algorithm is applied on numerical simulation and practical images. The quantitative analyzing results prove that...
The new modified anisotropic diffusion model can preserve textures effectively while ruling out the noise and the staircasing effects are decreased obviously. Compared with the original P–M model, more control parameters such as the standard deviation of Gaussian kernel $a$, property parameter of filter $h$ and non-local window size $K$ and $L$ are need to set in the NL P–M model, which make the calculating more complicated and time consuming. However, the iteration times decrease sharply. So in practical cases, we should take into account these factors to give suitable value to them, and sometimes, the experience is also useful to decide the close value of these parameters to get ideal results by less iteration times.
Fig. 2. Denoising results of cross-sectional image of a cigarette. (a) Original CT image, (b) denoising result by the P-M model, (c) denoising result by the NL-P-M model, (d) curves of the same line in the three images, (e) 3D virtual model based on the P-M method, and (f) 3D virtual model based on the NL-P-M method.

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